Systemic Risk in Financial Systems

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We consider default by firms that are part of a single clearing mechanism. The obligations of all firms within the system are determined simultaneously in a fashion consistent with the priority of debt claims and the limited liability of equity. We first show, via a fixed-point argument, that there always exists a "clearing payment vector" that clears the obligations of the members of the clearing system; under mild regularity conditions, this clearing vector is unique. Next, we develop an algorithm that both clears the financial system in a computationally efficient fashion and provides information on the systemic risk faced by the individual system firms. Finally, we produce qualitative comparative statics for financial systems. These comparative statics imply that, in contrast to single-firm results, even unsystematic, nondissipative shocks to the system will lower the total value of the system and may lower the value of the equity of some of the individual system firms.
(*Credit Risk; Default; Clearing Systems*)

1. Introduction

One of the most pervasive aspects of the contemporary financial environment is the rich network of interconnections among firms. Although financial liabilities owed by one firm to another are usually modeled as unidirectional obligations dependent only on the financial health of the issuing firm, in reality, the liability structure of corporate obligations is invariably much more intricate. The value of most firms is dependent on the payoffs they receive from their claims on other firms. The value of these claims depends, in turn, on the financial health of yet other firms in the system. Moreover, linkages between firms can be cyclical. A default by Firm A on its obligations to Firm B may lead B to default on its obligations to C. A default by C may, in turn, have a feedback effect on A. This example illustrates a general feature of financial system architectures, which we term *cyclical interdependence*. In this paper, we consider the problem of finding a clearing mechanism in cases in which this sort of cyclical interdependence is present.

All markets have some kind of clearing mechanism. Perhaps clearing mechanisms of interbank payments and for listed exchanges have received the most attention. In the United States, for example, CHIPS and Fedwire are the main banking clearing mechanisms; in Germany, the Abrechnung and the EAF (Elektronische Ai rechnung mit Filetransfer) performs this function. Regarding clearing mechanisms, one of the attractions of trading on a listed options exchange, the CBOE, for example, is that the Options Clearing Corporation is the counterparty to every trade. Hence, credit considerations do not prohibit lower-credit traders from participating in these markets. These payment systems are forced to confront large defaults on a regular basis. Examples of such defaults include the failure of I.D. Herstatt in 1974 and the Bank of New York overnight shortfall of $22.6 billion in 1985. Systemwide meltdowns also occur. For example, consider the collapse of the Tokyo real estate market, the bankruptcy and public bailout of American S&Ls to the cost of about $500 billion, the Venezuelan bank crisis of 1994, and the Long Term Capital bailout associated with the Russia's sovereign debt default. One of the most interesting failures of a tightly interconnected clearing system was the 1982 collapse of

0025-1909/01/4702/0236$5.00
1526-5501 electronic ISSN

the al-Manakh stock market in Kuwait. The clearing system, consisting of approximately 29,000 postdated checks written by traders, collapsed after a 45% drop in market values. The nominal gross liabilities of the participants in the market to each other at the time of the collapse was more than four times Kuwait's gross domestic product (Elimam et al. 1997).

Surprisingly, despite the obvious importance of the "architecture of financial linkages" for determining the return-generating process for financial assets, little has been written on cyclical financial interconnections. The effects of bilateral clearing of offsetting nominal obligations has been thoroughly analyzed in Duffie and Huang (1996). Rochet and Tirole (1996) analyzed the incentive and monitoring impact of an interbank loan. From a more empirical perspective, Angelini et al. (1996) develop an empirical model of intercorporate defaults. In this model, the probability that a default by one firm triggers another firm's default is exogenously specified without modeling intercorporate cash flows. Eliam et al. (1997) report the actual procedure used to clear intercorporate debts after the Kuwaiti shock market crash. However, to our knowledge, this paper is the first to analyze, in a general fashion, the properties of intercorporate cash flows in financial systems featuring cyclical interdependence and endogenously determined clearing vectors.

This lack of attention to cyclical interdependence is even more surprising given the extensive literature modeling default in a simple unidirectional and bilateral context. In fact, the whole literature on term structure of interest rates ignores the considerations mentioned above. While modeling the valuation of a firm's debt as independent from that of other firms simplifies debt and equity models, this assumption becomes questionable in portfolio management, corporate bond trading, and the analysis of counterparty credit risk. A desideratum for addressing these issues is the development of a simple, tractable model for computing clearing vectors for interlinked financial systems. The aim of this paper is to provide such a model.

We develop a fairly general model of a clearing system. The model satisfies the standard conditions imposed by bankruptcy law, that is, *clearing vectors*—which represent the vector of payments from nodes in the financial system to other nodes—satisfy the conditions of proportional repayments of liabilities in default, limited liability of equity, and absolute priority of debt over equity. We shall show, via a fixed-point argument, that clearing vectors always exist. Moreover, under mild regularity conditions, there is a unique clearing vector. This clearing vector can be computed through a "fictitious sequential default" algorithm. Moreover, the algorithm corresponds to a process of dynamic adjustment in which the set of defaulting firms at the start of each round is fixed by the adjustments of the system in the previous round. In each new round, an attempt is made to clear the system assuming that only nodes that defaulted in the last round default. If, in fact, no new defaults occur, the algorithm terminates. Otherwise, the new wave of defaults is recorded and the process is iterated again. This algorithm, as well as quickly yielding the clearing vector, produces a natural measure of *systemic risk*—the exposure of a given node in the system to defaults by other firms. This measure of systemic risk is based on how many "waves" of defaults are required to induce a given firm in the system to fail.

After analyzing the clearing vector, we perform comparative statics on the clearing payment vector, determining the nature of its dependence on the vector of operating cash flows as well as on the architecture of financial liabilities linking the various members of the system. More specifically, we show that the clearing payment vector is a multidimensional concave function of operating cash flows and the level of nominal payments, and that the value of equity is not generally convex in operating cash flows. These results imply that the total value of firms in the system is concave in operating cash flows. Standard results on stochastic dominance imply that the expectation of concave function or a random variable is lowered by increases in risk. Thus, our results imply, assuming standard risk-neutral valuation, that increased volatility, by lowering the value of interfirm payments, will lower the total value (debt plus equity) of nodes in the system. This result obtains even though there are no costs to insolvency in our model in the sense that total equity value is conserved. For this reason, our results suggest that using changes in total asset values to measure the effect of an economic shock on a group of tightly interconnected companies (e.g., Japanese banks) can be highly misleading.

The paper is organized as follows. In §2, we present the model and develop the basic machinery, including existence-uniqueness results. In §3, we present the two characterizations of the clearing vectors and examine their consequences. In §4, we derive comparative statics of the clearing system. Section 5 concludes the paper and considers some extensions.

2. Framework and Basic Results

2.1. Preliminaries

The subsequent analysis utilizes a number of standard definitions from matrix algebra and basic lattice theory. The definitions are standard. To reduce confusion and make referencing easier, we collect these definitions together in this section. Let $\Re^n$ represent n-dimensional Euclidean vector space. Let $\mathcal{N} = \{1, 2, \ldots, n\}$. For any two vectors, $x, y \in \Re^n$, define the lattice operations

$$x \wedge y := (\min[x_1, y_1], \quad \min[x_2, y_2] \ldots \min[x_n, y_n]),$$

$$x \vee y := (\max[x_1, y_1], \quad \max[x_2, y_2] \ldots \max[x_n, y_n]),$$

$$x^+ := (\max[x_1, 0], \quad \max[x_2, 0] \ldots \max[x_n, 0]).$$

Let $\mathbf{1}$ represent an n-dimensional vector, all of whose components equal 1, i.e., $\mathbf{1} = (1, \ldots, 1)$. Similarly, let $\mathbf{0}$ represent an n-dimensional vector, all of whose components equal 0. Let $\|\cdot\|$ denote the ℓ^1-norm on $\Re^n$. That is, for each $x \in \Re^n$ let

$$\|x\| := \sum_{i=1}^{n} |x_i|.$$

Let $|\|\cdot\||$ be the operator matrix Norm associated with $\|\cdot\|$; that is, for each $n \times n$ matrix, define

$$|\|M\|| \equiv \operatorname*{Sup}_{\|x\| \le 1} \|Mx\|.$$

An important definition for our future analysis is of a nonexpansive map. A map $T: \Re^n \to \Re^n$ is (ℓ^1)-*nonexpansive* if, $\forall x \in \Re^n$,

$$\|T(x) - T(y)\| \le \|x - y\|.$$

Whenever an ordering of elements of $\Re^n$ is specified in the sequel, the ordering refers to the pointwise ordering induced by the lattice operations, i.e.,

$$x \le y \Longleftrightarrow x_i \le y_i \quad \text{for all } i \in \mathcal{N}.$$

2.2. Economic Framework

Consider an economy populated by n nodes. Each of these nodes is to be thought of as a distinct economic entity, or *financial node*, participating in the clearing network. Each such entity may have nominal liabilities to other entities in the system. These nominal liabilities represent the promised payments due to other nodes in the system. We represent this structure of liabilities with the $n \times n$ *nominal liabilities matrix* L, where L_{ij} represents the nominal liability of node i to node j. As the notion of nominal claims seems to imply, we assume that all nominal claims are nonnegative and that no node has a nominal claim against itself. To reflect this economic interpretation, we specify that the nominal liabilities matrix is nonnegative and that all of the diagonal elements of the matrix equal 0; that is, we assume that $\forall i, j \in \mathcal{N}$, $L_{ij} \ge 0$ and that $\forall i$, $L_{ii} = 0$. Let $e_i \ge 0$ be the exogenous *operating cash flow* received by node i. This operating cash flow is the cash infusion to the node from sources outside the financial system. A *financial system* is thus a pair (L, e) consisting of a nominal obligations matrix L and an operating cash flow vector e, satisfying the conditions given above.

Note that the condition that operating cash flow is nonnegative is not really restrictive. It might appear that operating cash flows could be negative because of operating costs. However, operating costs are not negative cash infusions; rather, operating costs are the sum of all liabilities of the firm to outside factors of production: workers, suppliers, and so forth. A firm that has costs in excess of its revenues does not have a negative cash balance; rather, it has positive operating cash inflows and liabilities to workers and suppliers that exceed those positive operating cash flows. Those operating costs could be captured by appending to the financial system a "sink node," labeled, say, node 0. We could assume that this sink node has no operating cash flow of its own, i.e., $e_0 = 0$, nor obligations to other nodes, i.e., $L_{0j} = 0, \forall j$; the "operating cost" of node i, in this framework, would be the liabilities of node i to the sink node 0, i.e., L_{i0}. Because nothing in our setup precludes a node with the characteristics of the sink node, the assumption of nonnegative operating cash flows is made without a loss of generality.

Let p_i represent the total dollar payment by node i to the other nodes in the system. Let $p = (p_1, p_2, \ldots, p_n)$ represent the vector of total payments made by the nodes. Let $\bar{p}_i$ represent total nominal obligation of i to all other nodes, that is,

$$\bar{p}_i = \sum_{j=1}^{n} L_{ij}. \tag{1}$$

Let $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ represent the associated vector, which we term the *total obligation vector*. This vector represents the payment level required for complete satisfaction of all contractual liabilities by all nodes. Let

$$\Pi_{ij} \equiv \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and let Π represent the corresponding matrix, which we term the *relative liabilities matrix*. This matrix captures the nominal liability of one node to another in the system as a proportion of the debtor node's total liabilities. We assume that all debt claims have equal priority. This equality of priority implies that the payment made by node i to node j is given by $p_i\Pi_{ij}$. This implies that the total payments received by i are equal to $\sum_{j=1}^{n} \Pi_{ij}^T p_j$. Further, all payments are made to some node in the system, and, therefore,

$$\forall i, \sum_{j=1}^{n} \Pi_{ij} = 1,$$

or, in matrix notation,

$$\Pi \mathbf{1} = \mathbf{1},$$

an equality we will use later in the analysis.

The *total cash flow* to the owners of the equity of node i equals the sum of the payments received by other nodes plus the operating cash flow. This implies that the total cash flow to node i equals

$$\sum_{j=1}^{n} \Pi_{ij}^T p_j + e_i.$$

The *value of the equity of node i* is given by total cash flows less payments to creditors. In other words, the value of node i's equity is

$$\sum_{j=1}^{n} \Pi_{ij}^T p_j + e_i - p_i.$$

Note also that, by using (1) and (2), the financial system (L, e), where L is a nominal payments matrix and e is a vector of operating incomes, can be equivalently described by the corresponding triple $(\Pi, \bar{p}, e)$, where Π is a relative liabilities matrix, $\bar{p}$ is a payment vector, and e is an operating cash flow vector. We will flesh out this description of a financial system in the subsequent analysis.

Intuitively, a clearing payment vector for the financial system should represent a specification of the payments made by each of the nodes in the financial system that is consistent with the legal rules allocating value among nodes and among holders of debt and equity. Three criteria that must be satisfied are (a) limited liability, which requires that the total payments made by a node must never exceed the cash flow available to the node; (b) the priority of debt claims, which requires that stockholders in the node receive no value until the node is able to completely pay off all of its outstanding liabilities; and (c) proportionality, which requires that if default occurs, all claimant nodes are paid by the defaulting node in proportion to the size of their nominal claim on firm assets. These desiderata motivate the following definition.

Definition 1. A *clearing payment vector* for the financial system $(\Pi, \bar{p}, e)$ is a vector $p^* \in [\mathbf{0}, \bar{p}]$ that satisfies the following conditions:

a. *Limited Liability*. $\forall i \in \mathcal{N}$,

$$p_i^* \leq \sum_{j=1}^{n} \Pi_{ij}^T p_j^* + e_i.$$

b. *Absolute Priority*. $\forall i \in \mathcal{N}$, either obligations are paid in full, that is, $p_i^* = \bar{p}_i$, or all value is paid to creditors, that is,

$$p_i^* = \sum_{j=1}^{n} \Pi_{ij}^T p_j^* + e_i. \qquad \|$$

Under this definition, some firms will be forced to pay out all of their value to creditors. This fact raises the question of why firms facing certain default provide their cash flows to the clearing system, knowing that all cash they contribute will be paid out to other firms. We have in mind a situation in which ex ante there was uncertainty as to the realized cash flows of the firm. To raise funds for operations, firms borrowed from other firms in the network. Ex ante, firms expected to have positive equity balances in some states of nature. Ex post, uncertainty is resolved, and claims are cleared. It is this ex post clearing, corresponding to one of many realizations of the uncertainty faced ex ante, that we model in this paper. Ex post, some firms find themselves with zero equity balances, paying out all value to other firms in the system. Of course, ex post, firm owners would prefer not to make these payments. However, this is irrelevant because we assume a perfect claim-enforcement technology under which all ex ante commitments must be honored. The assumptions we make regarding contracting technology are entirely standard in the finance literature and adopted in countless articles. However, because bilateral clearing with a perfectly efficient contracting technology is a trivial problem, the extant literature places little emphasis on these assumptions. One central point of this paper is that the clearing problem is not trivial in a multilateral network with cyclical liabilities.

2.3. Existence of Clearing Payment Vectors

In the previous section we defined a clearing vector using the standard rules of value division between debtors and creditors: absolute priority, proportionality, and limited liability. In a context in which one firm is indebted to another firm, these rules always clearly specify a unique division of value between the debtor and creditor firms. Are these standard rules of value division sufficient to produce a unique division of value in a multifirm environment with cyclical obligations? Will there exist cases in which no division of value is consistent with these rules, or cases in which more than one division of value is consistent? We will show that a division of value consistent with standard rules of value division always exists. Moreover, under mild regularity conditions that ensure that all parties of the system actually have some value to distribute, only one pattern of payments is consistent with the standard rules of value division. In other words, we will show that clearing vectors exist and are unique.

To establish the existence of a clearing vector, we will require a fixed-point characterization of clearing vectors. To establish this fixed-point characterization, first note that limited liability and absolute priority imply that $p^* \in [\mathbf{0}, \bar{p}]$ is a clearing payment vector if and only if the following condition holds: $\forall i \in \mathcal{N}$,

$$p_i^* = \min\left[e_i + \sum_{j=1}^{n} \Pi_{ij}^T p_j^*, \bar{p}_i\right].$$

The first term in the minimum expression on the right-hand side of the above expression represents "what the node has," the total inflows to i. The second term in the minimum expression is "what the node owes," the total obligations of node i other nodes in the system. A clearing vector is a vector in which every node pays the minimum of what it has and what it owes. From the above discussion, we see that the clearing vector is a fixed point, p^*, of the map, $\Phi(\cdot\,; \Pi, \bar{p}, e)\colon [\mathbf{0}, \bar{p}] \to [\mathbf{0}, \bar{p}]$, defined by

$$\Phi(p; \Pi, \bar{p}, e) \equiv (\Pi^T p + e) \wedge \bar{p}.$$

An economic interpretation of Φ is that $\Phi(p)$ represents the total funds that will be applied to satisfy debt obligations, assuming that nodes receive inflows specified by p from their debt claims on other nodes. We now show, through a fixed-point argument using the Φ map, that every financial system has a clearing vector.

Theorem 1. *Corresponding to every financial system* $(\Pi, \bar{p}, e)$,

a. There exists a greatest and least clearing payment vector, p^+ *and* p^-.

b. Under all clearing vectors, the value of the equity at each node of the financial system is the same, that is, if p' *and* p'' *are any two clearing vectors,*

$$(\Pi^T(p') + e - \bar{p})^+ = (\Pi^T(p'') + e - \bar{p})^+.$$

Proof. To prove Theorem 1, we need to first characterize some basic properties of the Φ map. We note that Φ is positive, increasing, concave, and nonexpansive. The assertions of positivity, monotonicity, and concavity follow because Φ is the composition of the positive, increasing, affine map $q \to \Pi^T q + e$, and the positive, increasing, concave map $q \to q \wedge \bar{p}$. To show that the map is nonexpansive, first note that, for any three vectors x, y, and z, $\|x \wedge z - y \wedge z\| \leq \|x - y\|$. This result implies that $\|\Phi(p) - \Phi(p')\| = \|(\Pi^T p + e) \wedge \bar{p} - (\Pi^T p' + e) \wedge \bar{p}\| \leq \|\Pi^T p - \Pi^T p'\|$. Next note that the column sums of Π^T all equal 1. This implies, from basic matrix algebra, that $|||\Pi^T|||| = 1$. Thus, $\|\Pi^T p - \Pi^T p'\| \leq \|p - p'\|$, establishing nonexpansiveness.

Let FIX(Φ) represent the set of fixed points of Φ. Because Φ is increasing, $\Phi(\mathbf{0}) \geq \mathbf{0}$ and $\Phi(\bar{p}) \leq \bar{p}$, the Tarski fixed-point theorem (see, e.g., Zeidler 1986, Theorem 11.E) implies that FIX(Φ) is nonempty and, moreover, possesses a greatest and least element. Thus (a) is established.

To prove (b), let p' be any clearing vector. We will show that the value of equity is the same under p' and p^+. This is sufficient to establish (b). To show that the value of equity is the same under p' and p^+, first note that Π^T is an increasing map, as is the map $x \to x^+$. Thus, we must have, because $p^+ \geq p'$, that

$$(\Pi^T(p^+) + e - \bar{p})^+ \geq (\Pi^T(p') + e - \bar{p})^+.$$

Thus, if

$$(\Pi^T(p^+) + e - \bar{p})^+ \neq (\Pi^T(p') + e - \bar{p})^+,$$

then we would have that

$$(\Pi^T(p^+) + e - \bar{p})^+ \underset{\neq}{\geq} (\Pi^T(p') + e - \bar{p})^+. \tag{3}$$

Because p^+ and p^- are both clearing vectors, it also must be the case that

$$(\Pi^T(p^+) + e - \bar{p})^+ = \Pi^T(p^+) + e - p^+, \tag{4}$$

$$(\Pi^T(p') + e - \bar{p})^+ = \Pi^T(p') + e - p'. \tag{5}$$

Expressions (3), (4), and (5) imply that

$$\Pi^T(p^+) + e - p^+ \underset{\neq}{\geq} \Pi^T(p') + e - p'. \tag{6}$$

Now, note that $\Pi\mathbf{1} = \mathbf{1}$. This implies that

$$\mathbf{1} \cdot (\Pi^T(p^+) - p^+) = \mathbf{1} \cdot (\Pi^T(p') - p') = 0.$$

Thus,

$$\mathbf{1} \cdot (\Pi^T(p^+) + e - p^+) = \mathbf{1} \cdot (\Pi^T(p') + e - p'). \tag{7}$$

However, (6) implies that

$$\mathbf{1} \cdot (\Pi^T(p^+) + e - p^+) > \mathbf{1} \cdot (\Pi^T(p') + e - p'). \tag{8}$$

The contradiction between expressions (6) and (7) establishes (b). □

2.4. Uniqueness of Clearing Vectors

As we have seen, the existence of a clearing vector follows from a simple fixed-point argument. Establishing uniqueness for a large range of financial systems requires more work. We need to rule out cases where the same allocation of equity value can be supported by numerous specifications of payments between nodes. Cases exist in which clearing vectors are not unique. See Appendix 2 for an example. In this section we shall show that, to rule out such cases, we need only impose conditions that ensure that all parts of the system have some tangible economic value, in the form of operating cash flow, to distribute. To make these conditions precise, we require some definitions. The first key definition is that of a "surplus set."

Definition 2. A set $S \subset \mathcal{N}$ is a *surplus set* if no node in the set has any obligations to any node outside the set and the set has positive operating cash flows, that is, if $\forall(i, j) \in S \times S^c$, $\Pi_{ij} = 0$ and $\sum_{i \in S} e_i > 0$. ‖

Intuitively, a surplus set is a closed reservior of value in the financial system. Because the financial system is conservative, neither creating nor destroying value, the value in a surplus set must be allocated somewhere. Because the surplus set is closed, value must flow to some node in the surplus set itself. This observation is formalized in the next lemma.

Lemma 1. *If p^* is a clearing vector, then it is not possible for all nodes in a surplus set to have zero equity value.*

Proof. Suppose S is a surplus set. Let P_i^+ represent the sum of all of the payments received by a node $i \in S$ from nodes in S^c. By the definition of a surplus set, nodes in S make no payments to nodes in S^c. Thus, if all nodes in S have zero equity value, it must be the case that

$$p_i = \sum_{j \in S} \Pi_{ij}^T p_j + e_i + P_i^+, \qquad \forall i \in S. \tag{9}$$

Summing the equations specified in (9) over $i \in S$ thus yields

$$\sum_{i \in S} p_i = \sum_{j \in S} \sum_{i \in S} \Pi^T_{ij} p_j + \sum_{i \in S} (P^+_i + e_i). \tag{10}$$

Using the fact that S is a surplus set, we also have that

$$\sum_{i \in S} \Pi^T_{ij} = 1. \tag{11}$$

Expressions (10) and (11) imply that

$$0 = \sum_{i \in S} (P^+_i + e_i),$$

contradicting our assumption that $\sum_{i \in S} e_i > 0$. □

The second key to establishing uniqueness is a "financial structure graph," which describes in a qualitative fashion the links between the nodes in a financial system.

Definition 3. The *financial structure* graph associated with the financial structure $(\Pi, \bar{p}, e)$ is the directed graph whose vertices are the nodes of the financial system $\mathcal{N}$, and whose edges are defined by $i \to j \Longleftrightarrow \Pi_{ij} > 0$. ‖

The direct liabilities of each node in the system are to the nodes to which the agent has contractual obligations. However, these direct links by no means exhaust the set of all nodes that are affected by a node's default. Defaults cascade through the system. The default of a single node reduces the inflows to its creditors, perhaps triggering the default of one of these creditors, and even, perhaps, defaults further downstream. How far downstream can the risk of a given node in the system travel? An upper bound on propagation is provided by the concept of the *risk orbit* of a node in the system. The risk orbit of a node is the set of all nodes that are connected to the given node through some directed path, however circuitous, through the system.

Definition 4. For each node $i \in \mathcal{N}$, define the risk orbit of node i, denoted by $o(i)$, as follows: $o(i) = \{j \in \mathcal{N}\colon \text{there exists a directed path from } i \text{ to } j\}$.‖

It would appear that, because they abstract from the magnitude of exposures, concepts such as strong connectedness and risk orbits are incapable of providing any useful characterization of clearing payment vectors for the system. This is not correct. In fact, a very simple property of risk orbits forms the basis for our proof of the uniqueness of the clearing payment vector.

Lemma 2. *Suppose that p^* is a clearing vector for $(\Pi, \bar{p}, e)$. Let $o(i)$ be a risk orbit that satisfies $\sum_{j \in o(i)} e_j > 0$. Then, under p^*, at least one node of i has positive equity value, that is,*

$$\exists j \in o(i), \text{ such that } \bar{p}_j < (\Pi^T p^* + e)_j.$$

Proof. First note that $o(i)$ is a surplus set. To see this, note that if some node, say i', in $o(i)$ owed something to a node $j \in o(i)^c$, then, by appending to the directed path from i to i' the edge $i' \to j$, one could construct a directed path from i to j, contradicting the assumption that j is not in $o(i)$. Lemma 1 shows that every surplus set contains a node with positive equity value. □

The previous lemmas form the basis for a demonstration of the uniqueness of a clearing payment vector under a mild additional restriction that we term *regularity*.

Definition 5. A financial system is regular if every risk orbit, $o(i)$, is a surplus set.

Note that, in our model, real economic value is produced from operating income and this value is conserved by the clearing system. Bankrupt nodes have their value transferred to solvent creditor nodes. Moreover, our clearing system is closed; no value leaves the system. Regularity rules out cases where part of the network lacks any economic value, in the form of operating cash flows, to distribute. Thus, in essence, regularity boils down to the existence of some value somewhere in the system that can reach all points in the system. A simple sufficient condition for regularity is that all nodes have positive operating cash flows, another simple condition for regularity is that all nodes in the financial structure graph are strongly interconnected and some node has positive equity value. The next theorem shows that regularity is sufficient to ensure the existence of a unique clearing vector.

Theorem 2. *If the financial system is regular, the greatest and least clearing vectors are the same, i.e., $p^+ = p^-$, implying that the clearing vector is unique.*

Proof. See Appendix 1.

3. Characterizing Clearing Vectors

3.1. Sequence of Defaults

In this section we show that the clearing vector can be viewed as the product of a simulated or "fictitious" default process. This process both permits the construction of a simple algorithm for identifying clearing vectors and produces a natural metric for measuring a node's systemic risk exposure. We call this simple algorithm the *fictitious default algorithm*. The essence of the algorithm is simple. First, determine each node's payout, assuming that all other nodes satisfy their obligations. If, under the assumption that all nodes pay fully, it is, in fact, the case that all obligations are satisfied, then terminate the algorithm. If some nodes default even when all other nodes pay, try to solve the system again, assuming that only these "first-order" defaults occur. If only first-order defaults occur under the new clearing vector, then terminate the algorithm. If second-order defaults occur, then try to clear again assuming only second-order defaults occur, and so on. It is clear that since there are only n nodes, this process must terminate after n iterations. The point at which a node defaults under the algorithm is a measure of the node's exposure to the systemic risks faced by the clearing system.

We assume henceforth that the financial system has a unique clearing vector. As shown by Theorem 2, regularity is a sufficient condition for the clearing vector to be unique. In this section, we characterize this clearing vector. First we develop an algorithm of finding the clearing vectors. Describing the algorithm in detail and proving that it is effective requires that we develop some new concepts. Let $\overline{S}$ be the set of *supersolutions* of the fixed-point operator Φ; that is, $\overline{S} = \{p \in [\mathbf{0}, \bar{p}]\colon \Phi(p) \leq p\}$. The supersolutions are the set of proposed payment vectors under which payments received exceed payments required given the rules of limited liability and absolute priority. Thus, supersolutions are payment vectors under which some node is paying other nodes more than its total inflow. Note that, for any such supersolution, because total equity value is positive, it must be the case that not all nodes are paying more than their inflow, i.e., it is not possible that $\Phi(p) < \bar{p}$. For each $p \in \overline{S}$, let the default set under p, which we denote by $\mathbf{D}(p)$, be the set of nodes i, such that $\Phi(p)_i < \bar{p}_i$. By the earlier observation, $\mathbf{D}(p)$ cannot contain all nodes. Let $\Lambda(p)$ represent the $n \times n$ diagonal matrix defined as follows:

$$\Lambda(p)_{ij} = \begin{cases} 1 & i = j \text{ and } i \in \mathbf{D}(p) \\ 0 & \text{otherwise} \end{cases}.$$

$\Lambda(p)_{ij}$ is a diagonal matrix whose values equal 1 along the diagonal in those rows representing nodes not in default under p, and equal to 0 otherwise. Thus, when multiplied by other matrices or vectors, the Λ matrix converts the entries corresponding to the nondefaulting node to 0. The complementary matrix $I - \Lambda(p')$ converts entries corresponding to defaulting nodes to 0. For fixed $p' \in \overline{\mathbf{S}}$, define the map $p \to \mathrm{FF}_{p'}(p)$ as follows:

$$\begin{aligned} \mathrm{FF}_{p'}(p) \equiv{} & \Lambda(p')\big(\Pi^T(\Lambda(p')p + (I - \Lambda(p')\bar{p})) + e\big) \\ & + (I - \Lambda(p'))(\bar{p}). \end{aligned} \tag{FIX}$$

This map, $\mathrm{FF}_{p'}(p)$, simply returns, for all nodes not defaulting under p', the required payment $\bar{p}$, and, for all other nodes, returns the node's value assuming that nondefaulting nodes under p' pay in full and defaulting nodes under p' pay p. By our earlier result, Lemma 1, the default set is not a surplus set. Thus, $\Lambda(p)\Pi$ has a row sum that is less than 1, and no row sum exceeds 1; this, in turn, implies that $\mathrm{FF}_{p'}$ has a unique fixed point by standard input-output matrix results (Karlin 1959, Theorem 8.3.2). Call this fixed point $f(p')$. Note that only when p' is a supersolution can we be assured that $f(p')$ is well defined. Next, define inductively the following sequence of payment vectors:

$$p^0 = \bar{p}; \qquad p^j = f(p^{j-1}). \tag{FDS}$$

We call this sequence of vectors the *fictitious default sequence*, and we call the process of producing these vectors the *fictitious default algorithm*.

Lemma 3. *The fictitious default algorithm stated in* (*FDS*) *produces a well-defined sequence of vectors*, p^j. *This sequence decreases to the clearing vector in at most n iterations of the algorithm*.

Proof. First, we show by induction that the fictitious default sequence is well defined and decreasing. To show this, we must show that for all p^j, p^j is a supersolution to Φ for all j and that the sequence (p^j) decreases. We establish this result by induction. When $j = 0$, these assertions are obvious. Next, suppose the assertions are true for p^k. Note that the definition of the Λ matrix implies that $\Lambda(p^k)p^k + (I - \Lambda(p^k))\bar{p} = p^k$. Because p^k is a supersolution to Φ, it must be the case that for all defaulting nodes i, $(\Pi p^k + e)_i \leq p_i^k$. This implies, combined with the definition of Λ, that $\Phi(p^k) = \mathrm{FF}_{p^k}(p^k)$. By the induction hypothesis, p^k is a supersolution to Φ. Therefore, p^k is a supersolution to FF_{p^k}. This fact implies that p^{k+1}, the fixed point of FF_{p^k}, is less than or equal to p^k. Because $p^{k+1} \leq p^k$, the set of nodes at which default occurs must be no smaller under p^k than under p^{k+1}. Now, if the set of nodes is the same, then $\Phi(p^{k+1}) = \mathrm{FF}_{p^k}(p^k)$, which implies, because by definition p^{k+1} is a fixed point of $\mathrm{FF}_{p^k}(p^k)$, that p^{k+1} is a fixed point of Φ, and thus trivially a supersolution. If the set of defaulting nodes is larger under p^{k+1}, then some nodes that paid their obligations in full under p^k default under p^{k+1}, and the rest of the nodes either default under both payment vectors or under neither. Thus, for those nodes such that default occurs under p^{k+1} but not p^k, $\phi(p^{k+1})_i < p_i^{k+1}$. For all other nodes, the fixed-point construction implies that $\phi(p^{k+1})_i = p_i^{k+1}$. Thus, we have that p^j is a supersolution to Φ and that (p^j) is a weakly decreasing sequence.

As shown in the previous paragraph, if the set of defaulting nodes is the same under both p^{j+1} and p^j, then (i) p^j is a fixed point of Φ, and (ii) the sequence will remain constant after p^{j+1}. If p^j fails to be a fixed point of the map Φ, then a node that did not default under p^j defaults under p^{j+1}. In this case, the number of defaulting nodes, specified in the next Λ matrix, will increase in the next iteration. Because there are only n nodes and at most $n-1$ can default in any supersolution, it must be the case that the payment vector produced by the algorithm ceases to change after at most n iterations. Because the sequence is constant only at fixed points, the clearing vector is attained in at most n iterations. □

In addition to being computationally efficient, the algorithm has an economic interpretation: The step in the algorithm at which a node is added to the defaulting set can be used as a measure of the node's financial health. Nodes that default under the first trial solution are fundamentally insolvent because they cannot survive even with no systemic risk exposure. Nodes that fail in the next wave are quite fragile in that they fail whenever fundamentally insolvent nodes fail. The third-order failures are triggered by the failure of fragile, but not fundamentally unsound nodes, and so on. Thus, nodes are partitioned by the algorithm into solvent nodes and $1, 2 \ldots, n-1$th order failures.

3.2. Programming Characterization

Next we will show that clearing payment vectors can be identified by solving almost any programming problem that places weight on maximizing payments by all nodes in the system subject to the limited liability condition. Formally stated, we associate with each financial system $(\Pi, \bar{p}, e)$, and each function $f : [\mathbf{0}, \bar{p}] \to \Re$, the programming problem

$$\mathbf{P}(\Pi, \bar{p}, e, f) \qquad \max_{p \in [0, \bar{p}]} f(p)$$
$$s.t. \qquad p \leq \Pi^T p + e.$$

The link between this programming problem and clearing payment vectors for the financial system is provided by the following lemma.

Lemma 4. *If f is strictly increasing, then any solution to $\mathbf{P}(\Pi, \bar{p}, e, f)$ is a clearing vector for the financial system.*

Proof. If p^* solves $\mathbf{P}(\Pi, \bar{p}, e, f)$, the fact that p^* is a feasible solution to $\mathbf{P}(\Pi, \bar{p}, e, f)$ ensures that p^* satisfies the limited liability condition for a clearing payment vector. If absolute priority were not satisfied, say at node i, then it would be the case that $p_i^* < \bar{p}$ and

$$(\Pi^T p^* + e - p^*)_i > 0.$$

Consider the vector p_ϵ, which is equal to p^* in all components except i, and which, for i, is given by $p_i^* + \epsilon$, where ϵ is chosen sufficiently small to ensure that limited liability remains satisfied. Because

$$(\Pi^T p_\epsilon + e - p_\epsilon)_j - (\Pi^T p^* + e - p^*)_j = \epsilon \Pi_{ij} \geq 0,$$

p_ϵ is feasible. Because p_ϵ is at least equal to p^* in all its components and greater than p^* in one of its components, and because f is strictly increasing, it must be the case that $f(p^*) < f(p_\epsilon)$, contradicting the supposition that p^* is a solution to $\mathbf{P}(\Pi, \bar{p}, e, f)$. □

Because clearing vectors are determined entirely by the limited liability and absolute priority conditions, it follows that these two conditions always produce payoff vectors that maximize the total extraction of payments from the nodes in the financial system. Because the clearing vector is unique in any regular financial system, the result also implies that in regular financial systems, all decision makers who prefer more to less will agree that the clearing vector maximizes their objectives. Thus, for example, whether one attempts to maximize cents on the dollar paid or total payments, or payments weighted by a biased weighting scheme that favors some nodes over others, the end result will be the same—the selection of the clearing vector. The above result shows also that, for a regular financial system, solving the programming problem given by $\mathbf{P}(\Pi, \bar{p}, e, f)$ for a suitably chosen function f, say a linear function with positive weighting constants, is a way of computing the clearing vector. In fact, this is exactly the approach the monetary authorities in Kuwait took to clearing the financial net after the crash of the al-Manakh market. Given the $n-1$-step convergence of the fictitious default algorithm discussed above, however, this programming approach may not be an efficient way of computing clearing vectors, given that only one variable will be introduced into the basic solution on each pivot. Algorithms that exploit the economics of the problem, such as the fictitious default algorithm developed above, allow for the simultaneous introduction of many defaulting nodes in a single step.

4. The Comparative Statics of the Clearing System

The first question we will address is how this clearing payment vector changes with changes in the exogenous parameters of the model. We first consider the relationship between this clearing payment vector and the operating cash flows received by the system e, while holding the nominal liability matrix L (or equivalently Π and $\bar{p}$) constant. The basic characterization of this relationship is provided below.

LEMMA 5. *The clearing payment vector is a concave, increasing function of operating cash flow vector and the level of nominal liabilities. In other words, the function* $e \to \mathrm{FIX}(\Phi(\cdot; \Pi, \bar{p}, e))$, *and the function* $\bar{p} \to \mathrm{FIX}(\Phi(\cdot; \Pi, \bar{p}, e))$ *are concave, increasing, and nonexpansive.*

PROOF. For the purposes of this proof, define the function $F\colon [\mathbf{0}, \bar{p}] \times \Re^n_{++} :\to [\mathbf{0}, \bar{p}]$ by $F(p, e) \equiv \Phi(p, e; \Pi, \bar{p})$. The clearing payment vector is given by the function $f\colon \Re^n_{++} \to [\mathbf{0}, \bar{p}]$, defined by $f(e) = \mathrm{FIX}(F(\cdot, e))$. A theorem from Milgrom and Roberts (1994) shows that the fact that F is increasing in e (established in the proof of Theorem 1) implies that f is increasing. To see that f is concave and nonexpansive, define a sequence of functions, $\{f_n(e)\}_{n=o}^{\infty}$, inductively as follows:

$$f_n(e) = F(f_{n-1}(e), e), \qquad f_0(e) \equiv \mathbf{0}.$$

For each fixed $e \in \Re_{++}$, $f_n(e)$ is just the nth iteration of the map $p \to \Phi(p; \Pi, \bar{p}, e)$ function starting at the initial payment vector $\mathbf{0}$. Thus, standard results on the convergence of iterates of monotone increasing operators show that $f_n(e) \uparrow f(e)$, for all e. Using the fact that F is nondecreasing, jointly concave in p and e, and nonexpansive, induction shows that, for all n, f_n is concave and nonexpansive. Thus, f is the pointwise limit of nonexpansive concave functions, and thus concave and nonexpansive. The above argument establishes the claim of the lemma for the function $e \to \mathrm{FIX}(\Phi(\cdot; \Pi, \bar{p}, e))$. The proof of the claim for $\bar{p} \to \mathrm{FIX}(\Phi(\cdot; \Pi, \bar{p}, e))$ and $\Pi \to \mathrm{FIX}(\Phi(\cdot; \Pi, \bar{p}, e))$ is identical and thus will be omitted. □

Note that in the standard single-period/single-firm financial model, the payment to debtholders equals $\min[\bar{p}, e]$, where e is the firm's operating earnings and $\bar{p}$ is the level of the firm's nominal liabilities. Thus, the payment received by debtholders is a concave, increasing, nonexpansive function of the firm's operating cash flow and the level of nominal liabilities. Lemma 5 shows that these qualitative features of the debt payments in single-firm settings are inherited by the debt payment vectors of multinode clearing systems. This result has a number of direct implications. For example, suppose we allowed for stochastic operating cash flows. In this case, concavity of the payment stream in operating cash flows implies that increases in the riskiness of operating cash flows, in the sense of second-order stochastic dominance (Huang and Litzenberger 1988, Chap. 2), would reduce the expected payments on each debt claim. In other words, for all nodes i, $E[\tilde{p}_i]$ would fall with an increase in the risk of the operating cash flow vector, $\tilde{e}$. If we, in addition, imposed the standard assumptions for contingent claim pricing, e.g., that the financial markets are statically or dynamically complete, then the initial value of each node of the financial system would be given by its discounted expected value under the market pricing or "risk-neutral" probability measure (e.g., Duffie 1988, Chap. 22). Thus, our concavity result would imply, in this setting, that increases in risk under the pricing measure would lower the value of each traded debt claim.

The results for equity valuation are more interesting. The application of option pricing in the single-firm setting, as often taught in first-year finance courses, shows that equity may be priced as a call option on the value of the firm with the strike and maturity date equal, respectively, to the face value of 0-coupon debt and its maturity date. For the single firm, an increase in riskiness as represented by the volatility of the value of the firm (debt plus equity) not only decreases the value of debt, but also increases the value of equity. However, such risk shifts will not lead unambiguously to increased equity values for the nodes in a multifirm system. In a multifirm system, all debt claims are owned by stockholders at some nodes of the system. This implies that increases in risk across the system have two effects. First, they raise the value of equity by lowering the value of the debt payments made *to* other nodes. Second, the increased risk also lowers the value of payments *from* other nodes. Thus, the effect of risk increases on individual node equity is ambiguous. Because the total equity value of the system equals total operating cash flows, an increase in the volatility holding the mean constant has no effect on overall equity value. However, the lowered value of debt tends to reduce the value of the equity of those firms that are net creditors, and increase the value of the equity of net debtors.

Next, note that all of our results can also be interpreted in terms of node value. To understand this, note that the terminal-date equity in a financial system is $\Pi^T p^* + e - p^*$, and that the debt is $p^*(e)$, where p^* is the clearing vector for the financial system. Thus, the total terminal value of any node in the system is the value of debt plus the value of equity, or $\Pi^T p^* + e$. Total value of all nodes in the economy is thus just $\mathbf{1} \cdot (\Pi^T p^* + e) = \mathbf{1} \cdot (p^* + e)$, the sum of the value of equity and the value of all payments on liabilities under the equilibrium clearing vector.

Using this fact we can obtain another consequence of Lemma 5 that relates to the effect of cash flow volatility on the aggregate value of nodes in the financial system. Since, in an arbitrage-free economy, the value of a node is just the discounted expectation of its terminal value under the market-pricing measure, and because the function mapping cash flows to node value, $e \to \Pi^T p^*(e) + e$, is concave, increases in volatility, under the market-pricing measure, adversely affect firm value.

COROLLARY. *Increases in the volatility* (*under the market-pricing measure*) *of operating cash flows lowers the initial value of all nodes in the system.*

Thus, node value (debt plus equity) is reduced by economic volatility, even though, in our analysis, there are no dissipative consequences of financial distress even when markets are perfect and frictionless. Volatility reduces the size of payments between nodes, and this reduces the market value of nodes. Because, clearly, in the frictionless market setup specified above, volatility has no adverse overall welfare consequences, this result should be interpreted as a caution against interpreting the reduction in corporate value caused by risk as reflecting either market imperfections or irrational asset pricing.

Next, we show that, in some sense, convex combinations of financial systems can never have default or payment rates inferior to the worse of the two or superior to the better of the two. To permit a precise formulation of this idea, let $p^*(\Pi, \bar{p}, e)$ be the clearing payment vector associated with an arbitrary financial system $(\Pi, \bar{p}, e)$; that is, $p^*(\Pi, \bar{p}, e) \equiv \text{FIX}[\Phi(\cdot; \bar{p}, e)]$. A *$\lambda$-convex combination* of the financial systems $(\Pi', \bar{p}', e')$ and $(\Pi'', \bar{p}'', e'')$ is the financial system $(\Pi_\lambda, \bar{p}_\lambda, e_\lambda)$, defined by

$$(\Pi_\lambda, \bar{p}_\lambda, e_\lambda) = \lambda(\Pi', \bar{p}', e') + (1-\lambda)(\Pi'', \bar{p}'', e''),$$
$$\lambda \in [0, 1].$$

Lemma 6. *Suppose that the financial system* $(\Pi_\lambda, \bar{p}_\lambda, e_\lambda)$ *is a* λ*-convex combination of the financial systems* $(\Pi', \bar{p}', e')$ *and* $(\Pi'', \bar{p}'', e'')$*, then the equilibrium clearing payment vectors of the financial systems,* p^**, satisfy the following inequalities*:

$$p^*(\Pi', \bar{p}', e') \wedge p^*(\Pi'', \bar{p}'', e'') \le p^*(\Pi_\lambda, \bar{p}_\lambda, e_\lambda) \le p^*(\Pi', \bar{p}', e') \vee p^*(\Pi'', \bar{p}'', e'').$$

Proof. Note that, for all $i \in N$, the function $\lambda \to \Phi(p; \Pi_\lambda, \bar{p}_\lambda, e_\lambda)_i$ is linear, and therefore monotone. Thus we have that

$$\Phi(p; \Pi', \bar{p}', e') \wedge \Phi(p; \Pi'', \bar{p}', e'') \le \Phi(p; \Pi_\lambda, \bar{p}_\lambda, e_\lambda) \le \Phi(p; \Pi', \bar{p}', e') \vee \Phi(p; \Pi'', \bar{p}', e'').$$

Let

$$H^-(p) \equiv \Phi(p; \Pi', \bar{p}', e') \wedge \Phi(p; \Pi'', \bar{p}', e'');$$

$$H^+(p) \equiv \Phi(p; \Pi', \bar{p}', e') \vee \Phi(p; \Pi'', \bar{p}', e'').$$

Note that H^- and H^+ are monotone increasing maps defined on $[\mathbf{0}, \bar{p}]$ with fixed points in this order interval. If p^+ is a fixed point of H^+ and p^- is a fixed point of H^-, then the above inequality implies that

$$p^- \le p^*(\Pi_\lambda, e_\lambda) \le p^+.$$

Because $p^*(\Pi', \bar{p}', e') \vee p^*(\Pi'', \bar{p}'', e'')$ is a supersolution to H^+, i.e.,

$$p^+ \le p^*(\Pi', \bar{p}', e') \vee p^*(\Pi'', \bar{p}'', e''),$$

similarly, because $p^*(\Pi', \bar{p}', e') \wedge p^*(\Pi'', \bar{p}'', e'')$ is a subsolution to H^-,

$$p^- \ge p^*(\Pi', \bar{p}', e') \wedge p^*(\Pi'', \bar{p}'', e'').$$

The inequalities follow. □

Lemma 6 is a fairly weak result. However, a stronger characterization, such as a concavity result for financial systems (e.g., a result showing that convex combinations of systems yield higher payment rates than convex combinations of the payment vector of the two systems being combined), cannot be obtained. In fact, it is easy to construct counterexamples to this stronger characterization.[1] The failure of concavity occurs because the map $(\Pi, p) \to \Phi(p; \Pi, \bar{p}, e)$ is not concave, although it is concave in each of the variables, Π and p, separately.

[1] A numerical counterexample is available on request.

5. Possible Extensions and Concluding Remarks

In this paper, we provide conditions for the existence and uniqueness of a clearing vector for a complex financial system, analyze the properties of the clearing vector, and provide comparative statics describing the relationship between the clearing vector and underlying parameters of the financial system. This work represents a contribution to our understanding of the modeling of complex financial systems featuring cyclical obligations between the parties. However, it is only a first step in the development of a research program in this area. In fact, one of the virtues of our analysis is that it can be extended in many directions. Extensions fall into three broad categories: (i) utilizing the current model for valuation and risk analysis, (ii) dealing with more complex legal/institutional structures, and (iii) incorporating dynamics.

The simplest extension of the present analysis is to use the formulae developed in the paper to value financial claims and assess default probabilities for financial systems. Given a structure of liabilities, the value of the debt and equity claims for a fixed level of operating cash flows at the terminal date is determined by our model. If we assume operating cash flows follow a standard stochastic process between the initial date and the clearing date, then this stochastic process, combined with the terminal boundary conditions imposed by our model and standard risk-neutral valuation technology, can generate prices for the debt and equity of the nodes in the system (e.g., Duffie 1988). In addition, probabilities of default and default correlation can be computed easily. In addition, the distribution of cash flows to each of the nodes also can be computed and inverted to yield value-at-risk estimates.

Extending our results to allow for more complex legal and institutional structures is almost as transparent. For example, the nodes in the system could be allowed to hold intercorporate equity claims as well as intercorporate debt claims. In this case, inflows would be augmented by equity as well as debt inflows. Because equity claims are linear, this extension would not complicate our analysis significantly. Multiple priority classes could be accommodated by our framework. To accommodate multiple priority classes, we would utilize a sequential clearing procedure in which first a clearing vector for senior claims is found, then the residual value is treated as the operating cash flows of the system when clearing of the next highest priority claim, and so on. Another important extension would be to allow for violations of absolute priority, a significant factor in corporate bankruptcies, although not in some of the financial system clearing systems addressed earlier. The key assumptions that drive most of our results are that creditor claims are continuous and increasing in the value of the node. If violations of absolute priority are the product of efficient multilateral bargaining, as assumed in much of the literature (e.g., Brown 1989), then creditor claims are likely to have this property. In systems where there are substantial fixed costs of financial distress, continuity is lost and, for this reason, one would expect to obtain more opaque results: for example, the lack of a unique clearing vector even when mild regularity conditions, such as those used in this paper, are imposed.

The most difficult direction of extension would be to allow for more than one clearing date, and thus incorporate true dynamics. In principle the extension is straightforward and would proceed as follows. First, allow for intercorporate equity and assume that nodes that default at a given date become wholly owned by their creditors from that date forward. Next, allow all nodes to borrow from a node outside the system that itself is not subject to default risk. The outside node, or "central bank," would provide funds at a market-clearing rate. Thus, nodes would only default when, at the clearing vector, the value of future inflows is less than the value of liabilities. Using this motif and backward induction, one could recursively solve for clearing vectors. Uncertainty could be introduced into this framework by recursively computing the expected value of future inflows to determine the current economic value of the node and thus solve the default problem for successively earlier periods. Of course, this sort of extension of our analysis, through the "curse of dynamic programming," would greatly increase the complexity of the analysis.

Acknowledgment

The authors thank an anonymous referee for many helpful comments. In addition, they extend thanks to the participants in the 1999 Discrete Mathematics and Computer Science Conference on New Market Models for many helpful comments on an earlier draft of this paper. Comments by Isaac Sonin and Tomaz Slivnik were particularly appreciated. The usual disclaimer applies.

Appendix 1

Proof of Theorem 2. By Theorem 1, a greatest and least clearing vector exists. By definition, the greatest clearing vector, p^+, is at least weakly greater than the smallest clearing vector, p^-, i.e.,

$$p^+ \geq p^-. \tag{A1.1}$$

Suppose to obtain a contradiction that the greatest and least clearing vectors are unequal, i.e.,

$$p^+ \neq p^-. \tag{A1.2}$$

Let $E_j^+(E_j^-)$ represent the value of equity under clearing vector $p^+(p^-)$. Note that, by Theorem 1, the value of equity at all nodes is the same under all clearing vectors, i.e.,

$$\forall j, \quad E_j^+ = E_j^-. \tag{A1.3}$$

A straightforward consequence of (A1.3) is that the set of zero equity value nodes under p^+ equals the set of positive equity value nodes under p^-. Thus, without ambiguity we can apply the terms "zero equity value" and "positive equity value" to nodes without specifying the clearing vector.

By absolute priority, it must be the case that, for all nodes j that have positive equity value, $p_j^+ = p_j^- = \bar{p}_j$. Thus, if (A1.1) and (A1.2) hold, it must be the case that there exists a zero equity value node, i, such that

$$p_i^+ > p_i^-. \tag{A1.4}$$

Regularity means that the risk orbit of every node contains some node with a positive income. By the hypotheses of regularity and Lemma 3, the risk orbit of i must thus contain a positive equity value node. Thus, for some $l \in \{1, \ldots, n-1\}$, there exists a path

$$i = i_0 \to i_1 \to \cdots \to i_{l-1} \to i_l = m, \tag{A1.5}$$

where all nodes in the path are zero equity value nodes except for the last node, node m, and node m has positive equity value.

First we claim, by mathematical induction, that $p_{i_k}^+ - p_{i_k}^- > 0$ for nodes $i_0, \ldots i_k \cdots i_{l-1}$. The assertion is true by (A1.4) for $k = 0$. Now suppose the assertion is true at $k-1$. Because the nodes $i_0 \cdots i_{l-1}$ are zero equity value nodes, their payments equal their inflows. Thus for node i_k, $k \geq l-1$, it must be the case that

$$p_{ik}^+ = \sum_{i=1}^{n} \Pi_{ji_k} p_j^+ + e_{i_k} \quad \text{and} \quad p_{i_k}^- = \sum_{j=1}^{n} \Pi_{ji_k} p_j^- + e_{i_k}.$$

Thus,

$$p^+_{i_k} - p^-_{i_k} = \sum_{j=1}^{n} \Pi_{ji_k}(p^+_j - p^-_j). \tag{A1.6}$$

By the induction hypotheses $p^+_{i_{k-1}} - p^-_{i_{k-1}} > 0$. Because, $i_{k-1} \to i_k$, $\Pi_{i_{k-1}i_k} > 0$. Thus,

$$\Pi_{i_{k-1}i_k}(p^+_j - p^-_j) > 0. \tag{A1.7}$$

Expressions (A1.1), (A1.6), and (A1.7) show that $p^+_{i_k} - p^-_{i_k} > 0$. This result establishes the conclusion of the induction argument. This argument implies, in particular, that the last zero equity value node in the path, i_{l-1}, satisfies the conclusion of the argument, that is,

$$p^+_{i_{l-1}} - p^-_{i_{l-1}} > 0. \tag{A1.8}$$

Next, we show that (A1.8) implies that $E^+_m > E^-_m$. By the definition of equity value,

$$E^+_m - E^-_m = \sum_{j=1}^{n} \Pi_{jm}(p^+_j - p^-_j) - (p^+_m - p^-_m). \tag{A1.9}$$

Because m is a positive equity value node, absolute priority implies that $p^+_m = p^-_m = \bar{p}_m$; thus,

$$E^+_m - E^-_m = \sum_{j=1}^{n} \Pi_{jm}(p^+_j - p^-_j). \tag{A1.10}$$

Because, $i_{l-1} \to m$, $\Pi_{i_{l-1}m} > 0$. Thus,

$$\Pi_{i_{l-1}m}(p^+_{l-1} - p^-_{l-1}) > 0. \tag{A1.11}$$

Because, (A1.1), (A1.10), and (A1.11) hold, it must be the case that $E^+_m > E^-_m$. This assertion contradicts (A1.2), and this contradiction shows that the clearing vector must be unique. □

Appendix 2 Example of Nonuniqueness of the Clearing Vector in an Irregular Financial System

Some intuition for the importance of regularity for the uniqueness result is provided by the following example. Suppose the system contains two nodes, 1 and 2, both without any operating cash flows. Moreover, each node has nominal liabilities of 1.00 to the other node. In our notation we have that $e = (0, 0)^T$, $\bar{p} = (1, 1)$, and

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This system is not a regular financial system, because the single risk orbit of the system $\{1, 2\}$ is not a surplus set. In this example, any vector of the form $p_t = t(1, 1)$, $t \in [0, 1]$ is a clearing vector for the system. In contrast, if we modify the example by giving one cent to the first node by setting $e' = (0.01, 0)$, we see that the unique clearing vector is given by $p^* = (1.00, 1.00)$. The payment vectors p_t, $t < 1$, do not satisfy the absolute priority condition under given e' because they leave Node 1 with an equity balance of 0.01 despite the fact that Node 1 has not completely satisfied its nominal obligation to Node 2.

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Accepted by Phelim P. Boyle; received November 1999. This paper was with the authors 2 months for 2 revisions.